On the Multiplication of Successions of Fourier Constants.

By Prof. W. H. YOUNG, Sc.D., F.R.S.

(Received June 25,—Read June 27, 1912.)

§ 1. In a short note to appear in the 'Comptes Rendus' I have shown, by means of reasoning of a somewhat delicate nature, that, if we multiply together corresponding constants of two successions of Fourier constants, the succession of constants so obtained is the succession of the Fourier constants of a function whose summability depends on those of the two functions to which the two given successions belong in a certain definite way. If $(1+p)$ and $(1+q)$ represent the indices of the summability of the given functions, that of the new function so obtained is denoted by $(1+p)(1+q)/(1-pq)$. I have there shown how this theorem may be applied to obtain a notable extension of Parseval's theorem, and I have briefly indicated that we are thus able to give to that generalisation a still more complete form, namely, that the series

$$\sum_{n=1} (a_n^{2k}+b_n^{2k}),$$

where k is any positive integer, converges if the $2k/(2k-1)$th power of the function of which a_n and b_n are the Fourier constants is summable.

In the present communication I propose to give the necessary additional reasoning by which this result is obtained, and to deduce other important consequences. For this purpose we require the following generalisation of the first result noted above :—If the denominator of the expression

$$1+\mathrm{P} = \prod_{r=1}^{r=m}(1+p_r)/[1-\sum_{r=2}^{r=m}(r-1)\mathrm{T}_r],$$

where T_r denotes the sum of the products of the positive quantities $p_1, p_2, \ldots, p_m$, taken r at a time, is positive, then the succession of constants obtained by multiplying m successions of Fourier constants together is associated with a function whose index of summability is $1+\mathrm{P}$, when those of the separate functions are $1+p_r$, for all values of r from 1 to m.

In this statement the sine and cosine constants of the original functions may be interchanged an even number of times without affecting the truth of the statement.

When the p_r's are all equal, our formula takes the simple form

$$1+\mathrm{P} = (1+p)/[1-(m-1)p].$$

The limitation that the denominator has to be positive imposes a certain restriction in the enunciation of the theorems obtained. Thus it appears that, if $p < 1$, and a_n and b_n denote the typical Fourier constants of a function whose $(1+p)$th power is summable, the series

$$\sum_{n=1} n^{-q} a_n \quad \text{and} \quad \sum_{n=1} n^{-q} b_n \qquad [q > 1/(p+1)]$$

converge *absolutely*; this is an extension for the case in which $p < 1$ of a result I had already obtained by a different method in a communication already made to the Society, namely, that these series converge, whatever positive value be imputed to p. The absolute convergence in the case when $p < 1$ follows from the convergence, proved below, of the series

$$\sum_{n=1} n^{-q} (a_n^2 + b_n^2) \qquad [q > (1-p)/(1+p)].$$

As has been stated, our results have only been obtained on the supposition that p is less than unity. That the convergence of this latter series does not continue to subsist when p is greater than unity, follows from the examination of the Fourier series which defines Weierstrass's non-differentiable function. It appears, in fact, that continuous functions can be constructed whose Fourier constants, when raised to any positive power, however high, give a divergent series when multiplied by any power of n, however small. Thus we have no new theorems about the squares of the coefficients, if the function has more than its square summable; none about the fourth powers of the coefficients, if it has a power higher than 4/3 summable, and so on. Moreover, the remaining theorems of the type considered are certainly the most that can be obtained by the method here explained. In particular, we can only assert that the series whose general term is $n^{-t} a_n^m$, $\left(t > \dfrac{1-(m-1)p}{1+p}\right)$, converges, if $p \leq 1/(m-1)$, and can only assert that, when m is odd, the convergence is absolute, if $p \leq 1/(2m-1)$. It should also be remarked that we are only able to prove the former of these statements by applying the result of a paper presented some weeks ago to the Society.

When the denominator of the expression $1+\text{P}$ vanishes, it appears that the new function is a continuous function, so that its Fourier series converges when summed in the Cesàro manner. This is a remarkable extension of a result which I have already given for a pair of series.

This extension leads to the conclusion that the series $\sum\limits_{n=1} a_n^{2k+1}$ converges when the a_n's are positive, and, more generally, converges, whatever be the signs of the a_n's, when summed in the Cesàro manner, provided the $[(2k+1)/2k]$th power of the associated function is summable; and if, as appears probable, the $(1+p)$th power of the function associated with the allied series of a Fourier series is summable when a greater, or perhaps even an equal, power of the function associated with the Fourier series is summable, it leads to the result that the series $\sum\limits_{n=1} b_n^{2k+1}$ converges in the same manner, if the summability of the function is more than, or perhaps even equal to, $(2k+1)/2k$. It appears improbable that more than this can be stated. Indeed, when k is zero, it is not true that even $\sum\limits_{n=1} a_n$ converges in the ordinary manner, in the case in which the function has every power summable, or is even bounded. Still less is it true that $\sum\limits_{n=1} b_n$ converges.

There is an obvious direction in which those of our results which contain a power of n may be generalised. A slight modification may be shown to be allowable. We may, for example, for such an expression as n^{-q}, where $q > 1/(p+1)$, write $n^{1/(p+1)} g(n)$, where $g(n)$ is a suitable monotone function whose limit is zero. It would be foreign, however, to the purpose of the present paper to extend its results in this direction.

§ 2. In the note cited from the 'Comptes Rendus,' I show first that *if* $f(x)$ *and* $g(x)$ *are summable,* $\int f(x+t)\, g(t)\, dt$ *exists as an absolutely convergent integral, except for values of* x *forming a set of content zero, and represents a summable function of* x. I then prove that *the series*

$$\tfrac{1}{2}a_0 A_0 + \sum_{n=1} (a_n A_n + b_n B_n) \cos nx - \sum_{n=1} (a_n B_n - b_n A_n) \sin nx, \qquad \text{(C)}$$

where a_n, b_n *and* A_n, B_n *are the Fourier constants of the two functions* $f(x)$ and $g(x)$, *is the Fourier series of*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\, g(t)\, dt.$$

I then obtain the following inequality, where $0 \leq u$, $0 \leq v$, $0 < p$, $0 < q$, $pq < 1$,

$$\left(\int uv\, dx\right)^{(1+p)(1+q)/(1-pq)} \leq \left(\int v^{1+q} u^{1+p} dx\right) \left(\int u^{1+p} dx\right)^{q(1+p)/(1-pq)} \left(\int v^{1+q} dx\right)^{p(1+q)/(1-pq)}.$$

Putting $u = |f(x+t)|$, $v = |g(t)|$, and supposing the $(1+p)$th power of the former and the $(1+q)$th power of the latter to be summable, it follows from the first result quoted that the first integral on the right of the preceding inequality exists as an absolutely convergent integral, except for a set of values of x of content zero, and represents a summable function of x. Since the second and third integrals exist, the second being a bounded function of x, and the third independent of x, it follows that the left-hand side represents a summable function of x. In other words *the series* (C) *is the Fourier series of a function whose* $(1+p)(1+q)/(1-pq)$*th power is summable, provided* $f(x)$ *has its* $(1+p)$*th power and* $g(x)$ *its* $(1+q)$*th power summable, and* $pq < 1$.

§ 3. Hence by Parseval's theorem in its completed form, $\sum\limits_{n=1} a_n^2 A_n^2$ *converges if* $|f(x)|^{1+p}$ and $|g(x)|^{1+q}$ *are summable, provided*

$$(1+p)(1+q)/(1-pq) = 2. \tag{1}$$

Choosing $A_n = n^{-r-e}$, we may take

$$1+q = (1-r)^{-1} \qquad (0 < r < 1), \tag{2}$$

since the function corresponding to the Fourier series $\sum\limits_{n=1} n^{-r-e} \cos nx$ differs only by a continuous function from a constant multiple of x^{r+e-1}.

From (1) and (2) $\qquad (1+p)/(1-r-pr) = 2,$

and therefore $\qquad r = \frac{1}{2}\,\frac{1-p}{1+p} \qquad (0 < p < 1).$

Thus we have the following theorem :—

Theorem.—*If* a_n *is the typical Fourier cosine constant of a function* $f(x)$, *whose* $(1+p)$*th power is summable, the series* $\sum\limits_{n=1} n^{-t} a_n^2$ *converges provided*

$$t > \frac{1-p}{1+p} \qquad (0 < p < 1).$$

Under the same conditions $\sum\limits_{n=1} n^{-t} b_n^2$ *converges,* b_n *being the typical cosine constant of* $f(x)$, since the reasoning of the present article holds when we change $\cos nx$ into $\sin nx$.

This is an extension of the result given in § 15 of my paper "On a Mode of Generating Fourier Series," where the lower bound given for t was $1/(1+p)$.

§ 4. Hence we have also the following theorem :—

Theorem.—*If* a_n, b_n *are the typical Fourier constants of a function* $f(x)$ *whose* $(1+p)$*th power is summable where* $0 < p < 1$, *the series* $\sum\limits_{n=1} n^{-q} a_n$ *and* $\sum\limits_{n=1} n^{-q} b_n$ *converge absolutely, provided* $q > 1/(1+p)$.

For by the preceding theorem $\sum\limits_{n=1} \{n^{-t} a_n^2 + n^{-1-2e}\}$ converges, provided $1+t > 2/(1+p)$. Hence, for such values of t,

$$\sum_{n=1} n^{-\frac{1}{2}(1+t+2e)} |a_n| \qquad (0 > e)$$

converges, which proves the first statement. Similarly, the second statement follows, changing a_n into b_n.

I have already proved that *these series are convergent, though not that they are absolutely convergent, also for values of* $p \geq 1$, in a paper presented to the Society on May 11 of the present year, entitled "On the Convergence of Certain Series Involving the Fourier Constants of a Function."

§ 5. From what precedes, it follows that if $|f(x)|$ has its $(1+p)$th power summable and $|g(x)|$ its $(1+q)$th power summable, where $pq < 1$, the series

$$\tfrac{1}{2}a_0 A_0 + \sum_{n=1} a_n A_n \cos nx$$

is the Fourier series of a function whose $(1+p)(1+q)/(1-pq)$th power is summable. Hence we immediately get the result that, if in addition $h(x)$ has its $(1+r)$th power summable, the series

$$\tfrac{1}{2}a_0 A_0 \alpha_0 + \sum_{n=1} a_n A_n \alpha_n \cos nx,$$

where α_n is the typical Fourier cosine constant of $h(x)$, is the Fourier series of a function whose $(1+P_3)$th power is summable, where

$$1+P_3 = \frac{(1+p)(1+q)(1+r)}{1-pq-r(p+q+2pq)} = \frac{(1+p)(1+q)(1+r)}{1-T_2-2T_3},$$

T_k denoting the sum of the products of the indices $p, q, \ldots$ taken k at a time, and the denominator $1-T_2-2T_3$ is supposed positive.

We have accordingly the following theorem, which can be proved by induction:—

Theorem.—*If $f_r(x)$ has its $(1+p_r)$th power summable, for $r = 1, 2, \ldots, m$, and has for its typical Fourier cosine constant $a_{r,n}$ the series*

$$\tfrac{1}{2}a_{1,0}\, a_{2,0} \ldots a_{m,0} + \sum_{n=1} a_{1,n}\, a_{2,n} \ldots a_{m,n} \cos nx$$

is the Fourier series of a function whose $(1+P)$th power is summable, where

$$1+P = \frac{(1+p_1)(1+p_2)\ldots(1+p_m)}{1-T_2-2T_3-\ldots-(m-1)T_m},$$

T_k denoting the sum of the products of the quantities $p_1, p_2, \ldots p_m$, taken k at a time, provided the denominator of this fraction is positive.

In fact, since the numerator of the above fraction may be written

$$1+T_1+T_2+\ldots+T_m,$$

we have, assuming the theorem proved up to m and adding another function, for the index of the required power a fraction whose numerator evidently has the required form, and the denominator is

$$1-T_2-2T_3-\ldots-(m-1)T_m$$
$$-\{1+T_1+\ldots+T_m-1+T_2+2T_3+\ldots+(m-1)T_m\}p_{m+1},$$

which, when the products involving p_{m+1} are combined with the remaining products according to dimensions, takes the required form.

Since the statement has been proved for three functions, this proves the theorem.

Cor.—*If* $f(x)$ *is a function whose* $(1+p)$*th power is summable, where* $p<1/(m-1)$, *then the series* $\frac{1}{2}a_0{}^m+\sum_{n=1} a_n{}^m \cos nx$ *is the Fourier series of a function whose* $\left(\frac{1+p}{1-(m-1)p}\right)$*th power is summable.*

In fact, denoting the denominator of the fraction given in the enunciation of the theorem, when all the p_r's are equal, by $v(p)$, we have

$$v(p) = 1-\frac{m(m-1)}{1\,.\,2}p^2-\frac{m(m-1)(m-2)}{3\,!}2p^3-\dots-(m-1)p^m,$$

whence

$$p\int_0^p \frac{v(p)-1}{p^2}\,dp = 1+mp-(1+p)^m,$$

so that

$$v(p) = (1+p)^m-mp(1+p)^{m-1} = (1+p)^{m-1}\{1-(m-1)p\}.$$

Since the numerator is $(1+p)^m$, this gives the required result.

§ 6. We can now prove the following theorem :—

Theorem.—*If* $f(x)$ *is a function whose* $(1+p)$*th power is summable, where* $p=1/(m-1)$, *the series* $\frac{1}{2}a_0{}^m+\sum_{n=1} a_n{}^m \cos nx$ *is the Fourier series of a continuous function.*

For, writing $m=m_1+m_2$, we have $p<1/(m_1-1)$. Hence, by the preceding theorem, the series

$$\tfrac{1}{2}a_0{}^{m_1}+\sum_{n=1} a_n{}^{m_1}\cos nx \tag{i}$$

is the Fourier series of a function whose $(1+\text{M})$th power is summable,

where $$1+\text{M} = \frac{1+p}{1-(m_1-1)p} = \frac{1+p}{pm_2} = 1+\frac{1-(m_2-1)p}{pm_2}.$$

But $$\frac{1+p}{1-(m_2-1)p} = 1+\frac{pm_2}{1-(m_2-1)p} = 1+\frac{1}{\text{M}}.$$

Thus, by the same argument, the series (ii), got from (i) by changing m_1 into m_2, is the Fourier series of a function whose $(1+1/\text{M})$th power is summable.

Hence, by a known theorem,* the cosine series got from (i) and (ii) by multiplying together corresponding coefficients to form the coefficient of the corresponding term of the new series is the Fourier series of a continuous function. This proves the theorem.

As an immediate corollary we have the following theorem :—

Theorem.—*If* $f(x)$ *has its* $[m/(m-1)]$*th power summable,* $\Sigma a_n{}^m$ *converges when summed in the Cesàro manner. Hence, if* a_n *is positive for all values of* n, *or the integer* m *is even,* $\Sigma a_n{}^m$ *converges in the ordinary manner.*

---

* W. H. Young, "On a Class of Parametric Integrals and the Theory of Fourier Series," 'Roy. Soc. Proc.,' 1911, A, vol. 85, §§ 13 and 7.

§ 7. We have seen in § 5 that, if there are $(m+1)$ functions $f_r(x)$, the fraction giving the summability of the function corresponding to the Fourier series of cosines whose coefficients are the products of those in the Fourier series of the functions $f_r(x)$ may be written

$$\frac{(1+p_1)(1+p_2)\dots(1+p_m)(1+p_{m+1})}{1-T_2-2T_3-\dots-(m-1)T_m-(T_1+2T_2+3T_3+\dots+mT_m)p_{m+1}},$$

provided the denominator of this expression is positive. If, on the other hand, the denominator is zero, we have, with the notation of § 5,

$$p_{m+1} = P^{-1}.$$

Hence, by the same reasoning as in the preceding article, we have the following theorem, of which the first theorem of the preceding article is a particular case :—

Theorem.—*If* $f_r(x)$ *has its* $(1+p_r)$*th power summable, for* $r = 1, 2, \dots, m$, *and has for its typical Fourier cosine constant* $a_{r,n}$ *the series*

$$\tfrac{1}{2}a_{1,0}\,a_{2,0}\dots a_{m,0}+\sum_{n=1} a_{1,n}\,a_{2,n}\dots a_{m,n}\cos nx$$

is the Fourier series of a continuous function, provided the denominator of the fraction which, if finite, would give the summability of the function corresponding to the latter series, is zero, that is provided

$$0 = 1-T_2-2T_3-\dots-(m-1)T_m,$$

where T_r *denotes the sum of the products of the positive quantities* $p_1, p_2, \dots, p_m$, *taken* r *at a time.*

§ 8. If, as in § 3, we take

$$1+p_{m+1} = (1-r)^{-1},$$

$$a_{m+1,n} = n^{-r-e},$$

and, applying the work of the preceding article, put

$$p_{m+1} = -P^{-1},$$

we shall have

$$r = (1+P)^{-1}.$$

Hence we have the first result of the following theorem, using the result of § 7. The final statement follows from a theorem lately proved by myself in the paper already cited in § 4 :—

Theorem.—*If for* $r = 1, 2, \dots, m$, *the function* $f_r(x)$ *has its* $(1+p_r)$*th power summable, and has* $a_{r,n}$ *for its typical Fourier cosine constant, the series* $\sum\limits_{n=1} n^{-t}a_{1,n}a_{2,n}\dots a_{m,n}\cos nx$ *is the Fourier series of a continuous function, provided* $t > (1+P)^{-1} > 0$, *where* P *has the significance of* § 5.

Moreover, the series $\sum\limits_{n=1} n^{-t}a_{1,n}\,a_{2,m}\dots a_{m,n}$ then converges.

A special case is embodied in the following theorem:—

Theorem.—*If* $f(x)$ *has its* $(1+p)$*th power summable, and* a_n *is its typical Fourier cosine constant, the series*

$$\sum_{n=1} n^{-t} a_n^{\,m} \cos nx$$

is the Fourier series of a continuous function, provided

$$t > \frac{1-(m-1)p}{1+p} \quad \text{and} \quad p < \frac{1}{m-1},$$

and the series $\sum_{n=1} n^{-t} a_n^{\,m}$ *is then convergent.*

§ 9. *We can state more than the simple convergence of the series* $\sum_{n=1} n^{-t} a_n^{\,m}$, *if we further restrict* p *so as to be* $\leq 1/(2m-1)$, m being, of course, now an odd integer. *In fact the series in question is then absolutely convergent.*

When $p = 1/(2m-1)$, this follows from § 6, since the series

$$\sum_{n=1} (a_n^{\,2m} + n^{-1-2e})$$

is convergent, and therefore $\sum_{n=1} n^{-t} a_n^{\,m}$ converges absolutely for $t > \frac{1}{2}$.

When $p < 1/(2m-1)$, the result follows from the final theorem of the preceding article, since the series

$$\sum_{n=1} n^{-s-\frac{1}{2}-e} a_n^{\,m}$$

then converges absolutely, provided

$$s + \tfrac{1}{2} > \frac{1-(2m-1)p}{2(1+p)} + \tfrac{1}{2} > \frac{1-(m-1)p}{1+p}.$$

This proves the statement made at the beginning of the present article. Putting $m = 1$, we have the following special case:—

If $p \leq 1$, *the series* $\sum_{n=1} n^{-t} a_n$ *is absolutely convergent, and it is certainly convergent for all positive values of* p, t *being in all cases a quantity greater than* $\dfrac{1}{1+p}$.

Again, putting $m = 2$, we have the following:—

If $p < 1/3$, *the series* $\sum_{n=1} n^{-t} a_n^{\,2}$ *is absolutely convergent, and it is convergent if* $p < 1$; t *being in all cases any quantity greater than* $(1-p)/(1+p)$.

§ 10. Since $|f(x) - f(-x)|$ has the same degree of summability as $f(x)$, viz. the $(1+p)$th, and $g(x)$ has its $(1+q)$th power summable, the series

$$\tfrac{1}{2} b_0 B_0 + \sum_{n=1} b_n B_n \cos nx$$

is the Fourier series of a function whose $(1+p)(1+q)/(1-pq)$th power is summable. Hence it at once follows from the arguments used that *in our* *theorems we may replace an even number of the successions of cosine constants by sine constants.*

§ 11. If however in the series (C) we replace the a_n's by zeros and the B_n's by zeros, which, as we saw, is legitimate, we get a sine series, and not, as in the preceding article, a cosine series.

Thus the subsequent arguments used would not apply; they would of course apply if the a_n's and b_n's could be interchanged in the original series, that is if it is true that the allied series of the Fourier series of a function whose $(1+p)$th power is summable is also the Fourier series of a function whose $(1+p)$th power is summable. This proposition has not been proved, although, for various reasons of a more or less convincing character, I am inclined to believe that it is true, or at least that the allied series is the Fourier series of a function whose $(1+p_1)$th power is summable, where p_1 is any quantity less than p.

§ 12. In the various theorems here obtained there has always been a limitation on the value of p. Thus in the theorem concerning the convergence of $\Sigma n^{-t} a_n^2$, p had to be less than unity; if $p = 1$ we have the theorem of Parseval; in the theorem concerning the convergence of $\Sigma n^{-t} a_n^4$, p had to be less than $1/3$, while, if $p = 1/3$, the series Σa_n^4 is convergent, and so on. It should be noticed that, when p has a value greater than the limit imposed in the present paper, the index of the power to which n is raised changes sign. Our reasoning evidently fails in these cases. The question therefore remains over as to whether the formulæ are none the less true without the limitation imposed on p. That the answer to this question must be negative appears from a consideration of the series defining Weierstrass's non-differentiable function. This shows that we cannot even assert in the case of a continuous function that any positive power of n, however small, multiplied by any power of a_n, however large, need be the general term of a convergent series.

Indeed Weierstrass's series is

$$\sum_{n=1} a^n \cos b^n x,$$

where b is an odd integer, $a < 1$ and $ab > 1 + \frac{3}{2}\pi$. This is a Fourier series with groups of terms absent owing to the corresponding Fourier constants being zero. The series under consideration is therefore

$$\sum_{n=1} b^{nt} a^{nm} = \sum_{n=1} (b^t a^m)^n.$$

This will therefore certainly diverge if

$$b > a^{-m/t}.$$